风学数

命题解题

征解题 20260320

问题由 FMBS 数竞教练团队余智水老师提供.

问题

如图 1, Rt$\triangle ABC$ 中, $\angle B=90^\circ$, D 是 AB 边上一点, $AD=2BC$, 且 $\angle ACD=45^\circ$. 求 $\angle A$.

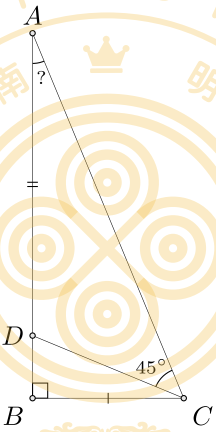

图 1

图南少培征解 欢迎提交解答

群聊: 图南少培

该二维码7天内(3月27日前)有效，重新进入将更新

解析

解法 1: (费振鹏) 结合基本形, 构造全等三角形.

如图 2, 过 A 作 $AE \perp CD$ 于 E, AE 与 CB 交于 F, 则 $AE = CE$. 又 $\angle EAD = \angle ECF$, 故

$$\text{Rt}\triangle ADE \cong \text{Rt}\triangle CFE.$$

所以 $AD = CF = 2BC$. 从而 $BF = BC$.

又 $AB \perp FC$, 故 $AF = AC$. 因此

$$\angle BAC = \frac{1}{2}\angle CAE = \frac{45^\circ}{2} = 22.5^\circ. \qquad \square$$

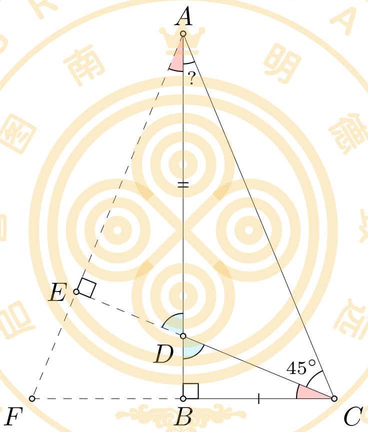

图 2

解法 2: (君成) 相似, 勾股定理.

如图 3.

作 $AE \perp CD$ 交 CD 延长线于 E, 则

$$AE = CE,\ \triangle AED \backsim \triangle CBD.$$

设 $DE = x$, $AE = y$, 则 $BD = \dfrac{x}{y}$, $CD = \dfrac{2}{y}$.

不妨设 $AD = 2$, $BC = 1$, 则

$$\begin{cases} x^2 + y^2 = 4, & ① \\ x + \dfrac{2}{y} = y. & ② \end{cases}$$

由 ② $\times 2y$ 得, $2xy + 4 = 2y^2$. 将 ① 代入得

$$(x+y)^2 = 2y^2.$$

$$x + y = \sqrt{2}y.$$

$$\frac{y}{x} = \sqrt{2} + 1.$$

代入 ② $\div x$ 得

$$1 + \frac{2}{yx} = \sqrt{2} + 1 \iff \frac{DC}{ED} = \frac{2}{yx} = \sqrt{2} = \frac{AC}{AE}.$$

所以 AD 为 $\angle EAC$ 的平分线.

因此 $\angle BAC = \dfrac{45^\circ}{2} = 22.5^\circ$. □

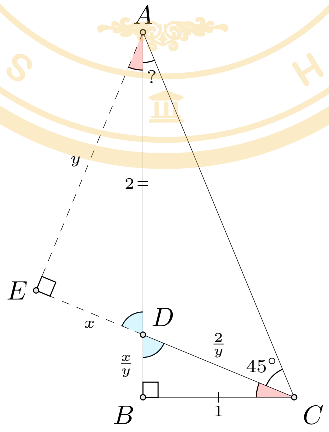

图 3

解法 3: (费振鹏) 分角定理 (面积法).

如图 4. 不妨设 $AD=2$, $BC=1$.

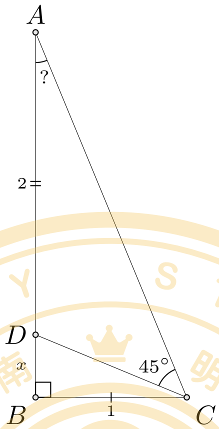

图 4

由 $\dfrac{S_{\triangle ACD}}{S_{\triangle BCD}}=\dfrac{AD}{BD}$, 得

$$\frac{AC^2\sin^2\angle ACD}{BC^2\sin^2\angle BCD}=\frac{AD^2}{BD^2}.$$

$$\frac{\left(x^2+4x+5\right)\cdot\frac{1}{2}}{\frac{x^2}{x^2+1}}=\frac{4}{x^2}.$$

$$\left(x^2+1\right)\left(x^2+4x+5\right)=8.$$

$$(x+1)^4=4.$$

而 $x+1>0$, 故 $x+1=\sqrt[4]{4}=\sqrt{2}$. 从而 $x=\sqrt{2}-1$.

所以 $\dfrac{BD}{BC}=\dfrac{\sqrt{2}-1}{1}=\dfrac{1}{\sqrt{2}+1}=\dfrac{BC}{AB}$.

即 $\tan\angle BCD=\tan\angle BAC$. 故

$$\angle BAC=\angle BCD=\frac{45^\circ}{2}=22.5^\circ.$$ □

解法 4: (颜芷媛) 正切的和角公式.

如图 5.

设 $BC = x$, $BD = y$, $\angle A = \alpha$, $\angle DCB = \beta$, 则

$$\alpha + \beta = 45^\circ.$$

$$\tan\alpha = \frac{x}{2x+y},\ \tan\beta = \frac{y}{x}.$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

所以 $x^2 - 2xy - y^2 = 0\,(x > y)$. 故 $x - y = \sqrt{2}y$.

$$\tan\beta = \frac{y}{x} = \sqrt{2} - 1.$$

$$\frac{1}{\tan\alpha} = 2 + \frac{y}{x} = 1 + \sqrt{2} \implies \tan\alpha = \sqrt{2} - 1.$$

所以 $\alpha = \beta \implies \alpha = \beta = 22.5^\circ$. □

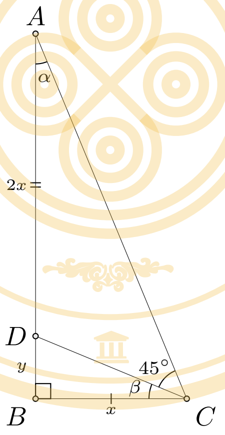

图 5

解法 5: (颜芷媛) 正弦定理, 积化和差.

如图 6.

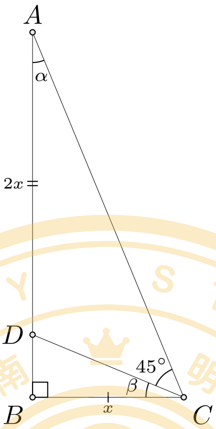

图 6

设 $BC = x$, $\angle A = \alpha$, $\angle DCB = \beta$.

在 $\triangle ACD$ 中, 由正弦定理, 得

$$\frac{2x}{\sin 45^\circ} = \frac{DC}{\sin \alpha} = \frac{\frac{x}{\cos \beta}}{\cos(\beta + 45^\circ)}.$$

$$2\cos\beta\cos(\beta + 45^\circ) = \sin 45^\circ.$$

$$\cos(2\beta + 45^\circ) + \cos 45^\circ = \sin 45^\circ.$$

$$\cos(2\beta + 45^\circ) = 0.$$

$$2\beta + 45^\circ = 90^\circ.$$

$$\beta = 22.5^\circ.$$

所以 $\alpha = 45^\circ - 22.5^\circ = 22.5^\circ$. □